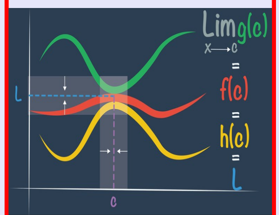
Calculus I

Lecture 16



Feb 19-8:47 AM

Prove if $\boxed{f(x) \text{ is differentiable at } a}$, then $\boxed{\text{it is continuous at } a}$.

Given $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

We need to show $\lim\limits_{x \to a} f(x) = f(a)$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left[f(x) - f(a) + f(a) \right]$$

$$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right]$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) + \lim_{x \to a} f(a)$$

$$= \boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \cdot \boxed{\lim_{x \to a} (x - a)} + \boxed{\lim_{x \to a} f(a)}$$

$$= f'(a) \cdot \underbrace{(a - a)}_{0} + f(a)$$

$$= f'(a) \cdot 0 + f(a) = f(a)$$

Since $\lim\limits_{x \to a} f(x) = f(a)$ then $f(x)$ is Cont. at a.

Jul 14-8:04 AM

Is $f(x)=\frac{x-3}{x+3}$ Cont. at $x=0$? Yes

Method I: Domain $(-\infty,-3)\cup(-3,\infty)$

$x+3\neq 0$

$x\neq -3$ So $x=0$ ✓

Method II: $f(x)=\frac{x-3}{x+3}$

$$f'(x)=\frac{1(x+3)-(x-3)\cdot 1}{(x+3)^2}=\frac{x+3-x+3}{(x+3)^2}=\frac{6}{(x+3)^2}$$

Is $f'(0)$ defined? $f'(0)=\frac{6}{(0+3)^2}=\frac{6}{9}=\frac{2}{3}$

Yes

$f(x)$ is differentiable at $x=0$

Since it is diff. at 0, it must be Cont. at 0.

Jul 14-8:13 AM

Evaluate $\lim\limits_{x\to\frac{\pi}{4}}\frac{\cos x-\frac{\sqrt{2}}{2}}{x-\frac{\pi}{4}}$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=-\sin\frac{\pi}{4}$$

$$=\boxed{-\frac{\sqrt{2}}{2}}$$

Jul 14-8:18 AM

Critical numbers:

Are the Solutions where $f'(x)=0$ or $f'(x)$ is undefined.

Critical points:

Are ordered-Pairs where $f(x)$ is evaluated at Critical numbers.

$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$

Domain $(-\infty,0)\cup(0,\infty)$

$f'(x) = \frac{2-x}{x^3}$

$f'(x)=0 \quad 2-x=0$ C.N. $\boxed{x=2}$

$f'(x)$ undefined C.N.
$x^3=0 \rightarrow \boxed{x=0}$ Not in the domain

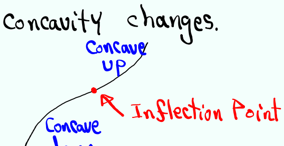
C.P. $(2, f(2)) \rightarrow (2, \frac{5}{4})$

$f(2) = 1 + \frac{1}{2} - \frac{1}{2^2} = 1 + \frac{1}{2} - \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}$

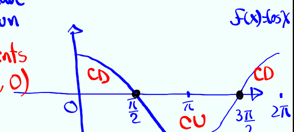
Jul 14-8:22 AM

Inflection points:

Are those points on the graph of $f(x)$ where Concavity changes.



Inflection Points $(\frac{\pi}{2},0), (\frac{3\pi}{2},0)$



How to find possible inflection Point:

Find Solutions where $f''(x)=0$ or $f''(x)$ is undefined

$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$

$(-\infty,0)\cup(0,\infty)$

$f''(x) = \frac{2(x-3)}{x^4}$

P. I. P.

$f''(x)=0 \quad 2(x-3)=0 \rightarrow \boxed{x=3}$

$f''(x)$ undef. $\quad x^4=0 \rightarrow \boxed{x=0}$ Not in the domain

$\rightarrow (3, \frac{11}{9})$

$f(3) = 1 + \frac{1}{3} - \frac{1}{3^2}$

$= 1 + \frac{1}{3} - \frac{1}{9}$

$= 1 + \frac{3}{9} - \frac{1}{9} = 1 + \frac{2}{9} = \frac{11}{9}$

Jul 14-8:29 AM

$f(x) = x^2 - 6x + 8$

Polynomial Function $\rightarrow$ Cont. $(-\infty, \infty)$

Y-Int $(0,8)$

$f(0) = 0^2 - 6(0) + 8 = 8$

x-Int $(2,0),(4,0)$

$f(x)=0$

$x^2 - 6x + 8 = 0$

$(x-2)(x-4)=0$

$x=2 \quad x=4$

$f'(x) = 2x - 6$

C.N.

$f'(x)=0 \rightarrow 2x-6=0 \quad \boxed{x=3}$

$f(3) = 3^2 - 6(3) + 8 = 9 - 18 + 8 = \boxed{-1}$

$\boxed{(3,-1)}$ C.P.

$f''(x) = 2 > 0$

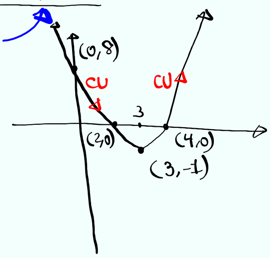
$f''(x) > 0 \rightarrow$ C.U.

$f''(x) < 0 \rightarrow$ C.D.

x	$-\infty$	3	∞
$f'(x)$	$-$	•	$+$
$f''(x)$	$+$		$+$
$f(x)$			

$f'(x) < 0$ Dec.

$f'(x) > 0$ Inc.



Jul 14-8:38 AM

$f(x) = x^4 - 4x^3$

1) Domain Polynomial Function $\rightarrow (-\infty, \infty)$

2) Y-Int $(0,0)$

3) x-Ints $(0,0),(4,0)$

$x^4 - 4x^3 = 0$

$x^3(x-4) \quad x=0$

$x=4$

4) $f'(x) = 4x^3 - 12x^2$

5) $f''(x) = 12x^2 - 24x$

6) Find C.P.

$f'(x)=0$

$4x^3 - 12x^2 = 0$

$4x^2(x-3)=0 \rightarrow 3^4 - 4 \cdot 3^3 = -27$

$x=0 \quad x=3$

$(0,0) \quad (3,-27)$

7) Find P.I.P.

$f''(x)=0$

$12x^2 - 24x = 0$

$12x(x-2) \quad 2^4 - 4 \cdot 2^3 = -16$

$x=0 \quad x=2$

$(0,0) \quad (2,-16)$

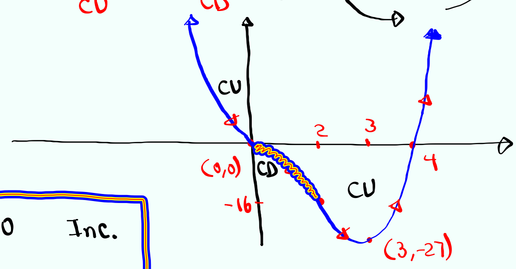
Jul 14-8:50 AM

8) Sign chart

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

x	$-\infty$	0		2		3	∞
$f'(x)$	$-$	•	$-$		$-$	•	$+$
$f''(x)$	$+$	•	$-$	•	$+$		$+$
$f(x)$	Dec. CU		Dec. CD		Dec. CU		Inc CU



$f'(x) > 0$ Inc.

$f'(x) < 0$ Dec.

$f''(x) > 0$ CU

$f''(x) < 0$ CD

Jul 14-9:00 AM

Looking Ahead:

$f'(x) = 2x + 8$ Find $f(x)$.

$\boxed{f(x) = x^2 + 8x + C}$

$f'(x) = 3x^2 - 2x + 8$, $\boxed{f(1) = 5}$ Find $f(x)$.

$f(x) = x^3 - x^2 + 8x + C$

$f(1) = \cancel{1^3} \cancel{-1^2} + 8(1) + C = 5$

$8 + C = 5$ $\boxed{C = -3}$

$\boxed{f(x) = x^3 - x^2 + 8x - 3}$

$f'(x) = \cos x - \sec^2 x$, $f(0) = 0$ find $f(x)$.

$f(x) = \sin x - \tan x + C$

$f(0) = \cancelto{0}{\sin 0} - \cancelto{0}{\tan 0} + C = 0$

$\boxed{C = 0}$

$\boxed{f(x) = \sin x - \tan x}$

Jul 14-9:12 AM

Rolle's Theorem

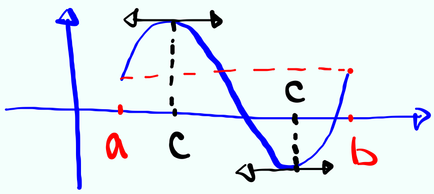
If $f(x)$ is Cont. on $[a,b]$,

$f(x)$ is differentiable on (a,b), and

$f(a) = f(b)$,

there is at least one number c in (a,b)

Such that $f'(c) = 0$.



Jul 14-9:44 AM

$f(x) = 3x^2 - 12x + 5 \quad , \quad [1,3]$

Polynomial function $\rightarrow$ Cont. $(-\infty, \infty)$

diff. $(-\infty, \infty)$

$f(1) = 3(1)^2 - 12(1) + 5 = 3 - 12 + 5 = -4$

$f(3) = 3(3)^2 - 12(3) + 5 = 27 - 36 + 5 = -4$

$\Rightarrow f(1) = f(3)$

By Rolle's thrm, there is at least one number c in $(1,3)$ such that $f'(c) = 0$

$f(x) = 3x^2 - 12x + 5$

$f'(x) = 6x - 12$

$f'(c) = 6c - 12 = 0$

$\boxed{c = 2}$

Yes

Jul 14-9:49 AM

Verify all Conditions of Rolle's thrm for $f(x) = x^3 - x^2 - 6x + 2$ over $[0,3]$, then find all numbers C for the Conclusion of Rolle's Thrm.

$f(x) = x^3 - x^2 - 6x + 2$

Polynomial

1) Cont. ✓ 2) Diff. ✓

$f(0) = 0^3 - 0^2 - 6(0) + 2 = 2$

$f(3) = 3^3 - 3^2 - 6(3) + 2 = 2$

3) $f(0) = f(3)$ ✓

Now Solve $f'(c) = 0$

$f'(x) = 3x^2 - 2x - 6$

$f'(c) = 3c^2 - 2c - 6$

$3c^2 - 2c - 6 = 0$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} = \frac{2 \pm \sqrt{76}}{6}$$

$(0,3)$

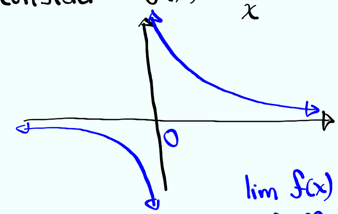
$\boxed{c = 1.782}$

$c = -1.120$

Google Mean-Value Theorem

Jul 14-9:54 AM

Consider $f(x) = \frac{1}{x}$



$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to 0^-} f(x) = -\infty$

$$\lim_{x \to \infty} \frac{2 + \frac{1}{x}}{5 - \frac{1}{x}} = \frac{\lim_{x \to \infty} \left(2 + \frac{1}{x}\right)}{\lim_{x \to \infty} \left(5 - \frac{1}{x}\right)} = \frac{\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x}}{\lim_{x \to \infty} 5 - \lim_{x \to \infty} \frac{1}{x}}$$

$$= \frac{2 + 0}{5 - 0} = \boxed{\frac{2}{5}} = \boxed{.4}$$

Use Your Calc

$$\frac{2 + \frac{1}{1000000}}{5 - \frac{1}{1000000}} \approx .40000028$$

Jul 14-10:05 AM

Evaluate $\lim_{x\to\infty} \frac{2x+1}{5x-1}$ $\frac{\infty}{\infty}$ I.F. $\rightarrow \frac{0}{0}, \frac{\infty}{\infty}$

Divide Top & Bottom by highest

Power of $x \Rightarrow x$

$$\lim_{x\to\infty} \frac{\frac{2x+1}{x}}{\frac{5x-1}{x}} = \lim_{x\to\infty} \frac{2+\cancel{\frac{1}{x}}\to 0}{5-\cancel{\frac{1}{x}}\to 0} = \boxed{\frac{2}{5}}$$

Jul 14-10:13 AM

Evaluate $\lim_{x\to\infty} \frac{1-\cancel{\frac{2}{x}}\to 0+\cancel{\frac{3}{x^2}}\to 0}{2+\cancel{\frac{3}{x}}\to 0-\cancel{\frac{2}{x^2}}\to 0}$

as $x\to\infty$

$\frac{1}{x}\to 0$

$\frac{1}{x^2}\to 0$

$= \boxed{\frac{1}{2}}$

Evaluate $\lim_{x\to\infty} \frac{x^2-2x+3}{2x^2+3x-2}$ $\frac{\infty}{\infty}$ I.F.

Divide by highest power of $x^{\boxed{2}}$

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}-\frac{2x}{x^2}+\frac{3}{x^2}}{\frac{2x^2}{x^2}+\frac{3x}{x^2}-\frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1-\cancel{\frac{2}{x}}\to 0+\cancel{\frac{3}{x^2}}\to 0}{2+\cancel{\frac{3}{x}}\to 0-\cancel{\frac{2}{x^2}}\to 0} = \boxed{\frac{1}{2}}$$

Jul 14-10:17 AM

Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x-5}{\sqrt{x^2+x}}$ $\quad \frac{\infty}{\infty}$ I.F.

$x\to\infty$

$x=\sqrt{x^2}$

Divide by $x^{\boxed{1}}$

$$=\lim_{x\to\infty} \frac{\frac{3x}{x}-\frac{5}{x}}{\frac{\sqrt{x^2+x}}{x}} = \lim_{x\to\infty} \frac{3-\frac{5}{x}}{\sqrt{\frac{x^2+x}{x^2}}}$$

$$=\lim_{x\to\infty} \frac{3-\cancel{\frac{5}{x}}\to 0}{\sqrt{1+\cancel{\frac{1}{x}}\to 0}} = \frac{3}{\sqrt{1}} = \boxed{3}$$

Jul 14-10:24 AM

Find two positive numbers whose product is 100, whose sum is a minimum

1 & 100 $\to$ Sum 101

2 & 50 $\to$ Sum 52

4 & 25 $\to$ Sum 29

10 & 10 $\to$ Sum 20

$x>0$, $y>0$

$xy=100 \to y=\frac{100}{x}$

$\underbrace{x+y}_{x+\frac{100}{x}}$ is minimum

$f(x)=x+\frac{100}{x}$

$f'(x)=1-\frac{100}{x^2}=\frac{x^2-100}{x^2}$

$f'(x)=0$
$x^2-100=0$ $\quad x=\pm 10$

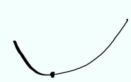
$f'(x)$ undefined $\quad x=0$

$f''(x)=\frac{200}{x^3}$

$f''(x)\neq 0$

$f''(x)$ undefined $\quad x=0$

x	0	10	∞
$f'(x)$	$-$	$\bullet$	$+$
$f''(x)$	$+$		$+$
$f(x)$	$\searrow$		$\nearrow$



(10, Minimum

$x=10$

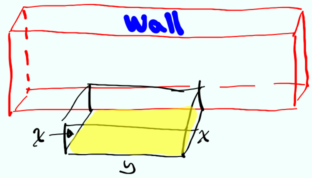
$y=\frac{100}{x}=\frac{100}{10}=10$

Two numbers are 10 & 10.

Jul 14-10:30 AM

I have 100 ft of fencing.

I want to make largest rectangular area next to a wall for our dog Toby to play.



$x + x + y = 100$

$2x + y = 100$

$y = 100 - 2x$

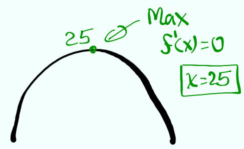
Largest area $\rightarrow A = xy$

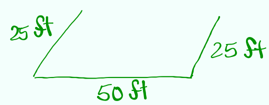
$= x(100 - 2x)$

$f(x) = x(100-2x)$ $\quad f(x) = 100x - 2x^2$

$f'(x) = 100 - 4x$

$f''(x) = -4 < 0$





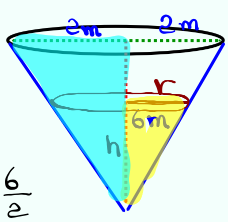
$2x + y = 100$

$y = 50$

Jul 14-10:43 AM

Water is leaking out of an inverted conical tank at the rate of 10000 cm^3/min.

$\frac{dV}{dt} = -10000\ cm^3/min$



$\frac{h}{r} = \frac{6}{2}$

$\frac{h}{r} = 3$ $\quad h = 3r$ $\quad r = \frac{h}{3}$

The tank has height of 6m and diameter at the top of 4m.

Find how fast the water level is decreasing when the height of water is 2m. 200 cm

Volume of a Cone

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h$ $\rightarrow V = \frac{\pi}{27} h^3$

$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2 \frac{dh}{dt}$

Jul 14-10:53 AM